

# $\kappa$ -Deformation of Poincaré Superalgebra with Classical Lorentz Subalgebra and its Graded Bicrossproduct Structure

*P. Kosiński<sup>\*¶</sup>, J. Lukierski<sup>†||</sup>, P. Maślanka<sup>‡¶</sup>, and J. Sobczyk<sup>§||</sup>*

## Abstract

The  $\kappa$ -deformed  $D = 4$  Poincaré superalgebra written in Hopf superalgebra form is transformed to the basis with classical Lorentz subalgebra generators. We show that in such a basis the  $\kappa$ -deformed  $D = 4$  Poincaré superalgebra can be written as graded bicrossproduct. We show that the  $\kappa$ -deformed  $D = 4$  superalgebra acts covariantly on  $\kappa$ -deformed chiral superspace.

---

<sup>\*</sup>Institute of Physics, University of Łódź, ul. Pomorska 149/153, 90-236 Łódź, Poland.

<sup>†</sup>SISSA, via Beirut 9, Trieste-Miramare, Italy, on leave of absence from the Institute for Theoretical Physics, University of Wrocław, pl. Maxa Born 9, 50-204 Wrocław, Poland.

<sup>‡</sup>Dept. of Functional Analysis, Institute of Mathematics, University of Łódź, ul. S. Banacha 22, 90-238 Łódź, Poland.

<sup>§</sup>International Centre for Theoretical Physics, 34100 Trieste, Italy, on leave of absence from Institute for Theoretical Physics, University of Wrocław, pl. Maxa Born 9, 50-204 Wrocław, Poland.

<sup>¶</sup>Partially supported by KBN grant 2P 302 21706.

<sup>||</sup>Partially supported by KBN grant 2P 302 08706.

# 1 Introduction

Following the formulation of  $D = 4$   $\kappa$ -deformed Poincaré algebra [1–3] recently also the  $\kappa$ -deformation of  $D = 4$   $N = 1$  Poincaré superalgebra was given [4]. Both deformations were firstly obtained in the framework of Hopf (super)algebras by the quantum contraction procedure of  $\mathcal{U}_q(O(3, 2))$  and  $\mathcal{U}_q(OSp(1|4))$  and have the following properties:

- a) The fourmomenta remain commutative, but noncocommutative,
- b) The three-dimensional rotations remain classical as Hopf algebras,
- c) The Lorentz generators do not form a subalgebra (neither the Lie subalgebra nor the Hopf subalgebra).

The first two properties imply that the deformation is "mild", and does not affect the rotational symmetry of nonrelativistic physics. The property c) is not convenient from physical point of view – in particular there are difficulties with the interpretation of finite  $\kappa$ -deformed Lorentz transformations which do not form a Lie group [5]. Recently, however, it has been given by Majid and Ruegg [6] the basis of quantum  $\kappa$ -Poincaré algebra, describing it as a bicrossproduct of classical Lorentz Hopf algebra  $O(3, 1)$  with the Hopf algebra  $T_4^\kappa$  of commuting fourmomenta equipped with  $\kappa$ -deformed coproduct

$$\mathcal{P}_4^\kappa = O(1, 3) \bowtie T_4^\kappa. \quad (1.1)$$

In such a framework the classical Lorentz algebra  $O(1, 3)$  (but not a classical Hopf algebra  $O(1, 3)$ !) is the subalgebra of  $\mathcal{P}_4^\kappa$ , and  $T_4^\kappa$  forms a Hopf subalgebra of  $\mathcal{P}_4^\kappa$ .

The aim of this paper is to find analogous basis for  $\kappa$ -deformed Poincaré superalgebra, with classical Lorentz subalgebra and commuting fourmomenta, which supersymmetrize the Majid–Ruegg bicrossproduct basis for  $\kappa$ -Poincaré algebra<sup>1</sup>. Such a formulation is derived (see Sect. 2) by nonlinear change of the basis of  $\kappa$ -Poincaré superalgebra, obtained previously in [4] from the quantum contraction of  $\mathcal{U}_q(OSp(1; 4))$ . It appears that the  $\kappa$ -Poincaré superalgebra  $\mathcal{P}_{4;1}^\kappa$  can be written e.g. as the following graded bicrossproduct<sup>2</sup> which extends supersymmetrically the formula (1.1):

$$\mathcal{P}_{4;1}^\kappa = O(1, 3; 2) \bowtie T_{4;2}^\kappa, \quad (1.2)$$

where  $O(1, 3; 2)$  is the classical superextension of the Lorentz algebra:  $(\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1))^3$ .

$$[M_{\mu\nu}^{(0)}, M_{\rho\tau}^{(0)}] = i(\eta_{\mu\tau}M_{\nu\rho}^{(0)} + \eta_{\nu\rho}M_{\mu\tau}^{(0)} - \eta_{\mu\rho}M_{\nu\tau}^{(0)} - \eta_{\nu\tau}M_{\mu\rho}^{(0)}) \quad (1.3)$$

---

<sup>1</sup>By supersymmetrization of  $\kappa$ -Poincaré algebra  $\mathcal{P}_4^\kappa$  we mean the  $\kappa$ -Poincaré superalgebra  $\mathcal{P}_{4;1}^\kappa$  which after formally putting in the bosonic sector the supercharges equal to zero reduces to  $\mathcal{P}_4^\kappa$ .

<sup>2</sup>We give here only one possibility – in fact there are four ways of expressing  $\mathcal{P}_{4;1}^\kappa$  as graded bicrossproduct (see Sect. 3).

<sup>3</sup>We denote by  $I_A^{(0)}$  the generators of classical Lie Hopf (super)algebras, with primitive coproducts  $\Delta(I_A^{(0)}) = 1 \otimes I_A^{(0)} + I_A^{(0)} \otimes 1$ .

with two complex supercharges  $Q_\alpha$  ( $\alpha = 1, 2$ ) satisfying the relations

$$[M_{\mu\nu}^{(0)}, Q_\alpha^{(0)}] = \frac{i}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta^{(0)}, \quad \{Q_\alpha^{(0)}, Q_\beta^{(0)}\} = 0, \quad (1.4a)$$

and  $\overline{T}_{4;2}^\kappa$  describes the complex Hopf superalgebra ( $\mu, \nu = 0, 1, 2, 3$ )

$$\{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0, \quad [\overline{Q}_{\dot{\alpha}}, P_\mu] = [P_\mu, P_\nu] = 0, \quad (1.4b)$$

supplemented by the following coproducts

$$\begin{aligned} \Delta P_i &= e^{-\frac{P_0}{\kappa}} \otimes P_i + P_i \otimes \mathbf{1}, \\ \Delta P_0 &= P_0 \otimes \mathbf{1} + \mathbf{1} \otimes P_0, \\ \Delta \overline{Q}_{\dot{\alpha}} &= \overline{Q}_{\dot{\alpha}} \otimes e^{-\frac{P_0}{2\kappa}} + \overline{Q}_{\dot{\alpha}} \otimes \mathbf{1}. \end{aligned} \quad (1.5)$$

In Sect. 3 we shall show that the  $\kappa$ -deformation of  $N = 1$  Poincaré superalgebra can be described by the  $\kappa$ -dependent action of the superalgebra  $O(1, 3; 2)$  on  $\overline{T}_{4;2}^\kappa$ , modifying in the algebraic sector the classical  $O(1, 3; 2)$ -covariance relations for the  $\overline{T}_{4;2}^\kappa$  generators, as well as  $\kappa$ -dependent coaction of  $\overline{T}_{4;2}^\kappa$  on  $O(1, 3; 2)$ , modifying the classical coproducts of the  $O(1, 3; 2)$  generators.

The bicrossproduct structure of  $\mathcal{P}_{4;1}^\kappa$  implies that the dual Hopf algebra  $(\mathcal{P}_{4;1}^\kappa)^*$  describing quantum  $N = 1$  Poincaré supergroup has also the bicrossproduct structure (see e.g. [8])

$$(\mathcal{P}_{4;1}^\kappa)^* = (O(1, 3; 2))^* \blacktriangleright (\overline{T}_{4;2}^\kappa)^*, \quad (1.6)$$

where  $(\overline{T}_{4;2}^\kappa)^*$  describes  $\kappa$ -deformed complex chiral superspace  $(x_\mu, \theta_{\dot{\alpha}})$  on which acts covariantly the  $\kappa$ -deformed superalgebra  $\mathcal{P}_{4;1}^\kappa$ . The  $\kappa$ -deformed superspace has been introduced recently in [7], but only the bicrossproduct structure of  $\mathcal{P}_{4;1}^\kappa$  permits to show that its chiral part transforms covariantly under the  $\kappa$ -deformed supersymmetry transformations. These transformations, obtained by the relation (1.2) are described in Sect. 4. In such a way we have all ingredients which are needed for the construction of  $\kappa$ -deformed chiral superfields formalism, what we shall present in our next publications.

## 2 The $\kappa$ -deformed $N = 1$ $D = 4$ Poincaré superalgebra with classical Lorentz generators.

Let us recall firstly the formulae describing the  $\kappa$ -deformed  $N = 1$   $D = 4$  Poincaré superalgebra as the noncommutative and noncocommutative real Hopf superalgebra. We have the following set of relations [4].

- a) Lorentz sector ( $M_{\mu\nu} = (M_i, N_i)$ , where  $M_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$  describe the non-relativistic  $O(3)$  rotations, and  $N_i = M_{0i}$  describe boosts).

i) *algebra* ( $\vec{P} = (P_1, P_2, P_3)$ ,  $\vec{M} = (M_1, M_2, M_3)$ )

$$[M_i, M_j] = i\epsilon_{ijk}M_k, \quad [M_i, L_j] = i\epsilon_{ijk}L_k \quad (2.1a)$$

$$[L_i, L_j] = -i\epsilon_{ijk} \left( M_k \cosh \frac{P_0}{\kappa} - \frac{1}{8\kappa} T_k \sinh \frac{P_0}{2\kappa} + \frac{1}{16\kappa^2} P_k (T_0 - 4\vec{P}\vec{M}) \right) \quad (2.1b)$$

where ( $\mu = 0, 1, 2, 3$ )

$$T_\mu = Q^A(\sigma_\mu)_{A\dot{B}}Q^{\dot{B}} \quad (2.2)$$

ii) *coalgebra*

$$\Delta(M_i) = M_i \otimes \mathbf{1} + \mathbf{1} \otimes M_i \quad (2.3a)$$

$$\begin{aligned} \Delta(L_i) = & L_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes L_i + \frac{1}{2\kappa} \epsilon_{ijk} (P_j \otimes M_k e^{\frac{P_0}{2\kappa}} + M_j e^{-\frac{P_0}{2\kappa}} \otimes P_k) \\ & + \frac{i}{8\kappa} (\sigma_i)_{\dot{\alpha}\beta} \left( \overline{Q}_{\dot{\alpha}} e^{-\frac{P_0}{4\kappa}} \otimes Q_{\beta} e^{\frac{P_0}{4\kappa}} + Q_{\beta} e^{-\frac{P_0}{4\kappa}} \otimes \overline{Q}_{\dot{\alpha}} e^{\frac{P_0}{4\kappa}} \right) \end{aligned} \quad (2.3b)$$

iii) *antipodes*

$$S(M_i) = -M_i, \quad (2.4)$$

$$S(N_i) = -N_i + \frac{3i}{2\kappa} P_i - \frac{i}{8\kappa} (Q\sigma_i\overline{Q} + \overline{Q}\sigma_i Q),$$

**b)** Fourmomenta sector  $P_\mu = (P_i, P_0)$  ( $\mu, \nu = 0, 1, 2, 3$ )

i) *algebra*

$$[M_i, P_j] = i\epsilon_{ijk}P_k, \quad [M_j, P_0] = 0, \quad (2.5a)$$

$$[L_i, P_j] = i\kappa\delta_{ij} \sinh \frac{P_0}{\kappa}, \quad [L_i, P_0] = iP_i, \quad (2.5b)$$

$$[P_\mu, P_\nu] = 0, \quad (2.5c)$$

ii) *coalgebra*

$$\Delta(P_i) = P_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes P_i, \quad (2.6a)$$

$$\Delta(P_0) = P_0 \otimes \mathbf{1} + \mathbf{1} \otimes P_0, \quad (2.6b)$$

The antipode is given by the relation  $S(P_\mu) = -P_\mu$ .

c) Supercharges sector [4]

i) *algebra*

$$\begin{aligned}\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} &= 4\kappa\delta_{\alpha\beta}\sin\frac{P_0}{2\kappa} - 2P_i(\sigma_i)_{\alpha\dot{\beta}}, \\ \{Q_\alpha, Q_\beta\} &= \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0,\end{aligned}\tag{2.7a}$$

$$[M_i, Q_\alpha] = -\frac{1}{2}(\sigma_i)_\alpha^\beta Q_\beta, \quad [M_i, \overline{Q}_{\dot{\alpha}}] = \frac{1}{2}(\sigma_i)_{\dot{\alpha}}^{\dot{\beta}} \overline{Q}_{\dot{\beta}},\tag{2.7b}$$

$$[L_i, Q_\alpha] = -\frac{i}{2}\cosh\frac{P_0}{2\kappa}(\sigma_i)_\alpha^\beta Q_\beta, \quad [L_i, \overline{Q}_{\dot{\alpha}}] = \frac{i}{2}\cosh\frac{P_0}{2\kappa}(\sigma_i)_{\dot{\alpha}}^{\dot{\beta}} \overline{Q}_{\dot{\beta}},\tag{2.7c}$$

$$[P_\mu, Q_\alpha] = [P_\mu, \overline{Q}_{\dot{\beta}}] = 0,\tag{2.7d}$$

ii) *coalgebra*

$$\begin{aligned}\Delta(Q_\alpha) &= Q_\alpha \otimes e^{\frac{P_0}{4\kappa}} + e^{-\frac{P_0}{4\kappa}} \otimes Q_\alpha, \\ \Delta(\overline{Q}_{\dot{\alpha}}) &= \overline{Q}_{\dot{\alpha}} \otimes e^{\frac{P_0}{4\kappa}} + e^{-\frac{P_0}{4\kappa}} \otimes \overline{Q}_{\dot{\alpha}},\end{aligned}\tag{2.8}$$

iii) *antipodes*

$$S(Q_\alpha) = -Q_\alpha, \quad S(\overline{Q}_{\dot{\alpha}}) = -\overline{Q}_{\dot{\alpha}}.\tag{2.9}$$

On the basis of the relations (2.3) - (2.7) one can single out the following features of the quantum superalgebra  $\mathcal{U}_\kappa(\mathcal{P}_{4;1})$  :

- i) The algebra coproducts and antipodes of Lorentz boosts  $N_i$  do depend on  $Q_\alpha, \overline{Q}_{\dot{\alpha}}$  i.e. the  $\kappa$ -deformed Poincaré as well as Lorentz sectors do not form the Hopf subalgebras.
- ii) Putting formally in the formulae (2.1) - (2.6)  $Q_\alpha = Q_{\dot{\alpha}} = 0$  one obtains the  $\kappa$ -deformed Poincaré algebra considered in [3], i.e.

$$\mathcal{U}_\kappa(\mathcal{P}_{4;1})|_{Q_\alpha=Q_{\dot{\alpha}}=0} = \mathcal{U}_\kappa(\mathcal{P}_4)\tag{2.10}$$

In order to remove from the formulae (2.1b) the supercharge-dependent terms one can introduce the following two complex Lorentz boosts

$$L_i^{(\pm)} = L_i \pm \frac{i}{8\kappa} T_i \quad (2.11)$$

complex-conjugated to each other  $((L_i^{(+)})^+ = L_i^{(-)})$ . One gets however

$$[L_i^{(\pm)}, L_j^{(\pm)}] = -i\epsilon_{ijk}(M_k \cosh \frac{P_0}{\kappa} - \frac{1}{4\kappa^2} P_k(\vec{P}\vec{M})) \quad (2.12)$$

Using (2.7 d) and (2.11) one obtains

$$[L_i^{(\pm)}, P_j] = i\kappa\delta_{ij} \sinh \frac{P_0}{\kappa}, \quad [L_i^{(\pm)}, P_0] = iP_i. \quad (2.13)$$

One can also calculate that

$$\begin{aligned} \Delta L_i^{(+)} = & L_i^{(+)} \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes L_i^{(+)} + \\ & + \frac{1}{2\kappa} \epsilon_{ijk} (P_j \otimes M_k e^{\frac{P_0}{2\kappa}} + M_j e^{-\frac{P_0}{2\kappa}} \otimes P_k) + \frac{i}{4\kappa} (\sigma_i)_{\dot{\alpha}\beta} e^{-\frac{P_0}{4\kappa}} \bar{Q}_{\dot{\alpha}} \otimes e^{\frac{P_0}{4\kappa}} Q_{\beta}, \end{aligned} \quad (2.14)$$

$$\begin{aligned} \Delta L_i^{(-)} = & L_i^{(-)} \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes L_i^{(-)} + \\ & + \frac{1}{2\kappa} \epsilon_{ijk} (P_j \otimes M_k e^{\frac{P_0}{2\kappa}} + M_j e^{-\frac{P_0}{2\kappa}} \otimes P_k) - \frac{i}{4\kappa} (\sigma_i)_{\dot{\alpha}\beta} e^{-\frac{P_0}{4\kappa}} Q_{\beta} \otimes e^{\frac{P_0}{4\kappa}} \bar{Q}_{\dot{\alpha}}, \end{aligned} \quad (2.15)$$

Further one can calculate the formulae in the supercharge sector. The modification (2.11) of the boost operators leads to the following covariance relations:

$$[L_i^{(\pm)}, Q_{\alpha}] = -\frac{i}{2} e^{\pm \frac{P_0}{2\kappa}} (\sigma_i Q)_{\alpha} \pm \frac{i}{4\kappa} P_i Q_{\alpha} \pm \frac{1}{4\kappa} \epsilon^{ijk} P_k (\sigma_i Q)_{\alpha} \quad (2.16a)$$

$$[L_i^{(\pm)}, \bar{Q}_{\dot{\alpha}}] = -\frac{i}{2} e^{\mp \frac{P_0}{2\kappa}} (\bar{Q} \sigma_i)_{\dot{\alpha}} \mp \frac{i}{4\kappa} P_i \bar{Q}_{\dot{\alpha}} \pm \frac{1}{4\kappa} \epsilon^{ijk} P_k (\bar{Q} \sigma_i)_{\dot{\alpha}} \quad (2.16b)$$

The relations (2.7a-b), (2.7d) and (2.16a-b) describe the supersymmetric extensions of the  $\kappa$ -Poincaré algebra  $\mathcal{U}_{\kappa}(\mathcal{P}_4)$ , given in [3].

In order to obtain the basis describing the bicrossproduct structure we introduce the following pairs of transformations

$$\begin{aligned} N_i^{(\pm)} &= \frac{1}{2} \{ L_i^{(\pm)}, e^{\mp \frac{P_0}{2\kappa}} \} \mp \frac{1}{2\kappa} \epsilon_{ijk} M_j P_k e^{\mp \frac{P_0}{2\kappa}}, \\ P_i^{(\pm)} &= P_i e^{\mp \frac{P_0}{2\kappa}}, \end{aligned} \quad (2.17)$$

$$Q_{\alpha}^{(\pm)} = e^{\mp \frac{P_0}{4\kappa}} Q_{\alpha}, \quad \bar{Q}_{\dot{\alpha}}^{(\pm)} = e^{\pm \frac{P_0}{4\kappa}} \bar{Q}_{\dot{\alpha}}. \quad (2.18)$$

It should be pointed out that for the generators  $(N_i^{(+)}, P_i^{(+)})$  the transformation (2.17) coincides with the one given in [6]. One obtains the following two Hopf superalgebra structures:

**a)** Lorentz sector  $(M_{\mu\nu}^{(\pm)} = (M_i, N_i^{(\pm)})$

*a1) algebra*

$$\begin{aligned} [M_i, M_j] &= i\epsilon_{ijk}M_k, \\ [M_i, N_k^{(\pm)}] &= i\epsilon_{ijk}N_k^{(\pm)}, \\ [N_i^{(\pm)}, N_j^{(\pm)}] &= -i\epsilon_{ijk}M_k, \end{aligned} \tag{2.19}$$

*a2) coalgebra*

$$\Delta(M_i) = M_i \otimes \mathbf{1} + \mathbf{1} \otimes M_i, \tag{2.20a}$$

$$\begin{aligned} \Delta(N_i^{(+)}) &= N_i^{(+)} \otimes \mathbf{1} + e^{-\frac{P_0}{\kappa}} \otimes N_i^{(+)} + \\ &+ \frac{1}{\kappa}\epsilon_{ijk}P_j^{(+)} \otimes M_k + \frac{i}{4\kappa}(\sigma_i)_{\dot{\alpha}\beta}e^{-\frac{P_0}{\kappa}}\overline{Q}_{\dot{\alpha}}^{(+)} \otimes Q_{\beta}^{(+)}, \end{aligned} \tag{2.20b}$$

$$\begin{aligned} \Delta(N_i^{(-)}) &= N_i^{(-)} \otimes e^{\frac{P_0}{\kappa}} + \mathbf{1} \otimes N_i^{(-)} \\ &- \frac{1}{\kappa}\epsilon_{ijk}M_k \otimes P_j^{(-)} - \frac{i}{4\kappa}(\sigma_i)_{\dot{\alpha}\beta}Q_{\beta}^{(-)} \otimes e^{\frac{P_0}{\kappa}}\overline{Q}_{\dot{\alpha}}^{(-)}, \end{aligned} \tag{2.20c}$$

*a3) antipode*  $(T_i \equiv T_i^{(+)} = T_i^{(-)})$

$$\begin{aligned} S(M_i) &= -M_i, \\ S(N_i^{(\pm)}) &= -\left[ N_i^{(\pm)} \pm \frac{1}{\kappa}\epsilon_{ijk}M_jP_k^{(\pm)} \right. \\ &\quad \left. \mp \frac{3i}{2\kappa}P_i^{(\pm)} \pm \frac{i}{4\kappa}\left(2P_i^{(\pm)} - T_i e^{\mp \frac{P_0}{2\kappa}}\right) \right] e^{\pm \frac{P_0}{\kappa}}, \end{aligned} \tag{2.21}$$

**b)** Fourmomentum sector:  $P_{\mu}^{(\pm)} = (P_i^{(\pm)}, P_0^{(\pm)} = P_0)$

*b1) algebra*

$$[P_{\mu}^{(\pm)}, P_{\nu}^{(\pm)}] = 0, \tag{2.22a}$$

$$[M_i, P_j^{(\pm)}] = i\epsilon_{ijk}P_k^{(\pm)}, \quad [M_i, P_0] = 0, \tag{2.22b}$$

$$[N_i^{(\pm)}, P_j^{(\pm)}] = \pm i \delta_{ij} \left[ \frac{\kappa}{2} (1 - e^{\mp \frac{2P_0}{\kappa}}) + \frac{1}{2\kappa} \vec{P}^{(\pm)2} \right] \mp \frac{1}{\kappa} P_i^{(\pm)} P_j^{(\pm)}, \quad (2.22c)$$

$$[N_i^{(\pm)}, P_0] = i P_i^{(\pm)}, \quad (2.22d)$$

*b2) coalgebra*

$$\Delta P_0 = P_0 \otimes \mathbf{1} + \mathbf{1} \otimes P_0, \quad (2.23)$$

$$\Delta P_i^{(+)} = P_i^{(+)} \otimes \mathbf{1} + e^{-\frac{P_0}{\kappa}} \otimes P_i^{(+)}, \quad (2.24)$$

$$\Delta P_i^{(-)} = P_i^{(-)} \otimes e^{\frac{P_0}{\kappa}} + \mathbf{1} \otimes P_i^{(-)},$$

*b3) antipode*

$$S(P_i^{(\pm)}) = -e^{\pm \frac{P_0}{\kappa}} P_i^{(\pm)}, \quad S(P_0) = -P_0, \quad (2.25)$$

**c) Supercharge sector**  $(Q_\alpha^{(\pm)}, \overline{Q}_\alpha^{(\pm)})$

*c1) algebra*

$$\begin{aligned} [M_i, Q_\alpha^{(\pm)}] &= -\frac{1}{2} (\sigma_i Q^{(\pm)})_\alpha \\ [M_i, \overline{Q}_{\dot{\alpha}}^{(\pm)}] &= \frac{1}{2} (\overline{Q}^{(\pm)} \sigma_i)_{\dot{\alpha}} \end{aligned} \quad (2.26)$$

$$\begin{aligned} [N_i^{(+)}, Q_\alpha^{(+)}] &= -\frac{i}{2} (\sigma_i Q^{(+)})_\alpha, \\ [N_i^{(+)}, \overline{Q}_{\dot{\alpha}}^{(+)}] &= -\frac{i}{2} e^{-\frac{P_0}{\kappa}} (\overline{Q}^{(+)} \sigma_i)_{\dot{\alpha}} + \frac{1}{2\kappa} \epsilon_{ikl} P_k^{(+)} (\overline{Q}^{(+)} \sigma_l)_{\dot{\alpha}}, \\ [N_i^{(-)}, Q_\alpha^{(-)}] &= -\frac{i}{2} (\sigma_i Q^{(-)})_\alpha, \\ [N_i^{(-)}, \overline{Q}_{\dot{\alpha}}^{(-)}] &= -\frac{i}{2} e^{\frac{P_0}{\kappa}} (\overline{Q}^{(-)} \sigma_i)_{\dot{\alpha}} - \frac{1}{2\kappa} \epsilon_{ikl} P_k^{(-)} (\overline{Q}^{(-)} \sigma_l)_{\dot{\alpha}}, \end{aligned} \quad (2.27)$$

$$[Q_\alpha^{(\pm)}, P_\mu^{(\pm)}] = [\overline{Q}_{\dot{\alpha}}^{(\pm)}, P_\mu^{(\pm)}] = 0, \quad (2.28)$$

$$\{Q_\alpha^{(\pm)}, \overline{Q}_{\dot{\beta}}^{(\pm)}\} = 4\kappa \delta_{\alpha\dot{\beta}} \sinh\left(\frac{P_0}{2\kappa}\right) - 2e^{\pm \frac{P_0}{2\kappa}} P_i^{(\pm)} (\sigma_i)_{\alpha\dot{\beta}}, \quad (2.29)$$

*c2) coalgebra*

$$\Delta(Q_\alpha^{(+)}) = e^{-\frac{P_0}{2\kappa}} \otimes Q_\alpha^{(+)} + Q_\alpha^{(+)} \otimes \mathbf{1}, \quad (2.30)$$

$$\Delta(\overline{Q}_\alpha^{(+)}) = \mathbf{1} \otimes \overline{Q}_\alpha^{(+)} + \overline{Q}_\alpha^{(+)} \otimes e^{\frac{P_0}{2\kappa}}, \quad (2.31)$$

$$\Delta(Q_\alpha^{(-)}) = \mathbf{1} \otimes Q_\alpha^{(-)} + Q_\alpha^{(-)} \otimes e^{\frac{P_0}{2\kappa}}, \quad (2.32)$$

$$\Delta(\overline{Q}_\alpha^{(-)}) = e^{-\frac{P_0}{2\kappa}} \otimes \overline{Q}_\alpha^{(-)} + \overline{Q}_\alpha^{(-)} \otimes \mathbf{1}, \quad (2.33)$$



*c3) antipodes*

$$S(Q_\alpha^{(\pm)}) = -Q_\alpha^{(\pm)} e^{\mp \frac{P_0}{2\kappa}}, \quad S(\overline{Q}_\alpha^{(\pm)}) = -\overline{Q}_\alpha^{(\pm)} e^{\mp \frac{P_0}{2\kappa}}, \quad (2.34)$$

We see that we obtain two  $\kappa$ -deformed Poincaré Hopf superalgebras (see also (2.15);  $R = 1, \dots, 14$ )

$$\mathcal{U}_\kappa^{(\pm)}(\mathcal{P}_{4;1}) : \tilde{I}_R^{(\pm)} = (M_i, N_i^{(\pm)}, P_i^{(\pm)}, P_0, Q_\alpha^{(\pm)}, \overline{Q}_\alpha^{(\pm)}), \quad \tilde{I}_R^{(+)} = \left(\tilde{I}_R^{(-)}\right)^\oplus. \quad (2.35)$$

related by the nonstandard involution  $\oplus$  (see also [8]) introduced by the change of sign of  $\kappa$  i.e. satisfying the following properties:

$$(ab)^\oplus = a^\oplus b^\oplus, \quad \kappa^\oplus = -\kappa, \quad (2.36)$$

$$(a \otimes b)^\oplus = b^\oplus \otimes a^\oplus.$$

We see therefore that we obtained two supersymmetric extensions of  $\kappa$ -Poincaré algebra written in bicrossproduct basis. In the following section we shall describe the graded bicrossproduct structure.

### 3 Graded bicrossproduct structure of $\kappa$ -deformed Poincaré superalgebra

Let us write the classical  $N = 1$  Poincaré superalgebra as the following graded semidirect product

$$\mathcal{P}_{4;1} = O(1, 3; 2) \bowtie \overline{T}_{4;2}, \quad (3.1)$$

where the superalgebras  $SO(1, 3; 2)$  and  $\overline{T}_{4;2}$  are given respectively by the formulae (1.3), (1.4a) and (1.4b). The cross relations describing  $O(1, 3; 2)$  covariance are

$$[M_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu), \quad (3.2a)$$

$$[Q_\alpha, P_\rho] = 0, \quad (3.2b)$$

$$[M_{\mu\nu}, \overline{Q}_{\dot{\alpha}}] = \frac{i}{2}(\sigma_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} \overline{Q}_{\dot{\beta}} \quad (3.2c)$$

$$\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2(\sigma_\mu)_{\alpha\dot{\beta}} P^\mu. \quad (3.2d)$$

We see that the basic superalgebra relation (3.2d) occurs as a covariance relation.

The graded semidirect product formula (3.1) can also be written as follows:

$$\mathcal{P}_{4;1} = \overline{O(1, 3; 2)} \bowtie T_{4;2}, \quad (3.3)$$

where  $\overline{O(1, 3; 2)} = (M_{\mu\nu}, \overline{Q}_{\dot{\alpha}})$  and  $T_{4;2} = (P_{\mu}, Q_{\alpha})$ . It is easy to see that the relations (3.2a-3.2d) also describe the cross relations for (3.3), with the role of  $Q_{\alpha}$  and  $\overline{Q}_{\dot{\alpha}}$  in the bicrossproduct interchanged.

The  $\kappa$ -deformed Poincaré superalgebras (2.35) can be treated as  $\kappa$ -deformations of the semidirect products (3.1), taking the form of the bicrossproducts<sup>4</sup>. We obtain

$$\mathcal{U}_{\kappa}^{(+)}(\mathcal{P}_{4;1}) = T_{4;2}^{\kappa(+)} \blacktriangleright O(1, 3; 2), \quad (3.4a)$$

$$\mathcal{U}_{\kappa}^{(-)}(\mathcal{P}_{4;1}) = O(1, 3; 2) \blacktriangleleft T_{4;2}^{\kappa(-)}, \quad (3.4b)$$

where  $O(1, 3; 2) = (M_{\mu\nu}^{(0)}, Q_{\alpha}^{(0)})$  and  $T_{4;2}^{\kappa(\pm)} = (P_{\mu}, \overline{Q}_{\dot{\alpha}}^{(\pm)})$ . We see from the relation (2.23) and (2.30) – (2.33) that the Hopf superalgebras  $T_{4;2}^{\kappa(\pm)}$  are described by the classical superalgebra and noncocommutative coproducts; the Hopf superalgebra  $O(1, 3; 2)$  is classical in the algebra as well as the coalgebra sectors. The bicrossproduct structure of (3.4a-b) is described by

i) The actions  $\hat{\alpha}^{(\pm)}$

$$\hat{\alpha}^{(+)} : T_{4;2}^{\kappa(+)} \otimes O(1, 3; 2) \longrightarrow T_{4;2}^{\kappa(+)}, \quad (3.5a)$$

$$\hat{\alpha}^{(-)} : O(1, 3; 2) \otimes T_{4;2}^{\kappa(-)} \longrightarrow T_{4;2}^{\kappa(-)}, \quad (3.5b)$$

modifying the cross relations (3.2a-3.2d),

ii) The coactions  $\hat{\beta}^{(\pm)}$

$$\hat{\beta}^{(+)} : O(1, 3; 2) \longrightarrow T_{4;2}^{\kappa(+)} \otimes O(1, 3; 2), \quad (3.6a)$$

$$\hat{\beta}^{(-)} : O(1, 3; 2) \longrightarrow O(1, 3; 2) \otimes T_{4;2}^{\kappa(-)}, \quad (3.6b)$$

modifying the classical coproducts for the generators of  $O(1, 3; 2)$ .

Further we shall consider only the bicrossproduct described by the action  $\hat{\alpha}^{(+)}$  and coaction  $\hat{\beta}^{(+)}$ . From the formulae (2.22c) and (2.27-2.29) it is easy to check that  $\hat{\alpha}^{(+)}$  has the following nonlinear (i.e.  $\kappa$ -deformed) components<sup>5</sup>

$$\begin{aligned} \hat{\alpha}^{(+)}(P_j^{(+)} \otimes N_i^{(+)}) &= -i\delta_{ij} \left[ \frac{\kappa}{2}(1 - e^{-\frac{2P_0}{\kappa}}) + \frac{1}{2\kappa}\vec{P}^{(+)^2} \right] + \frac{i}{\kappa}P_i^{(+)}P_j^{(+)}, \\ \hat{\alpha}^{(+)}(\overline{Q}_{\dot{\alpha}}^{(+)} \otimes N_i^{(+)}) &= \frac{i}{2}e^{-\frac{P_0}{\kappa}}(\overline{Q}^{(+)}\sigma_i)_{\dot{\alpha}} - \frac{1}{2\kappa}\epsilon_{ikl}P_k^{(+)}(\overline{Q}^{(+)}\sigma_i)_{\dot{\alpha}}, \\ \hat{\alpha}^{(+)}(\overline{Q}_{\dot{\beta}}^{(+)} \otimes Q_{\alpha}^{(+)}) &= 4\kappa\delta_{\alpha\dot{\beta}}\sinh\frac{P_0}{2\kappa} - 2e^{\frac{P_0}{2\kappa}}P_i^{(+)}(\sigma_i)_{\alpha\dot{\beta}}. \end{aligned} \quad (3.7)$$

<sup>4</sup> The bicrossproducts of Hopf algebras were introduced by Majid ([8]; see also [10]). The notion of crossproducts for braided quantum groups, which can be considered the generalization of the notion of quantum supergroups, was discussed in [11].

<sup>5</sup>The components which are not deformed describe standard semidirect product.

Similarly, it follows from the formulae (2.20b) and (2.30 – 2.33) that the coaction  $\widehat{\beta}^{(+)}$  has the following  $\kappa$ -dependent components<sup>6</sup>

$$\begin{aligned}\widehat{\beta}^{(+)} \left( N_i^{(+)} \right) &= e^{-\frac{P_0}{\kappa}} \otimes N_i^{(+)} + \frac{1}{\kappa} \epsilon_{ijk} P_j^{(+)} \otimes M_k + \frac{i}{4\kappa} (\sigma_i)_{\alpha\dot{\beta}} e^{-\frac{P_0}{\kappa}} \overline{Q}_{\dot{\alpha}}^{(+)} \otimes Q_{\beta}^{(+)} , \\ \widehat{\beta}^{(+)} \left( Q_{\alpha}^{(+)} \right) &= e^{-\frac{P_0}{2\kappa}} \otimes Q_{\alpha}^{(+)} .\end{aligned}\tag{3.8}$$

It can be checked that the actions (3.7) and coactions (3.8) satisfy the axioms required by the bicrossproducts structure [10, 11].

The action  $\widehat{\alpha}^{(-)}$  and coaction  $\widehat{\beta}^{(-)}$  can be obtained from  $\widehat{\alpha}^{(+)}$  and  $\widehat{\beta}^{(+)}$  by the nonstandard involution (2.36), changing the sign of the parameter  $\kappa$  as well as the order in the tensor product.

Finally one can show that modifying properly the definitions (2.17) – (2.18) of the bicrossproduct basis one can introduce the quantum  $N = 1$  Poincaré superalgebra as the  $\kappa$ -deformed crossproduct (3.3) supplemented by suitably deformed crosscoproduct. In this way we obtain two other ways of expressing quantum deformation of  $\mathcal{P}_{4;1}$  in the bicrossproduct form.

## 4 $\kappa$ -deformed covariant chiral superspace

Following the general theory of bicrossproducts [8, 10, 11] from the formula (3.4) follows that the quantum  $\kappa$ -deformed  $N = 1$  supergroup  $(\mathcal{P}_{4;1}^{\kappa})^*$  dual to the  $\kappa$ -deformed Poincaré superalgebra (3.4a-b) can be written also in the bicrossproduct form (see e.g. (1.6))<sup>7</sup>, where

- $(O(1, 3; 2))^* = C(SO(1, 3; 2))$  is the graded commutative algebra of functions on the graded Lorentz group  $SO(1, 3; 2)$ .
- $(T_{4;2}^{\kappa(\pm)})^*$  is the graded algebra of functions on  $\kappa$ -deformed chiral superspace, dual to the Hopf algebras  $T_{4;2}^{\kappa(\pm)}$ .

The action and coaction which describes the bicrossproduct Hopf structure of the deformed graded algebra of functions on  $N = 1$  Poincaré supergroup can be obtained from the actions and coactions (3.7 – 3.8) by respective dualizations.

Let us describe firstly the  $\kappa$ -deformed chiral superspace. Using the relations

$$\begin{aligned}\langle t, zz' \rangle &= (-1)^{\eta(z)\eta(t_{(2)})} \langle t_{(1)}, z \rangle \langle t_{(2)}, z' \rangle , \\ \langle tt', z \rangle &= (-1)^{\eta(z_{(1)})\eta(t')} \langle t, z_{(1)} \rangle \langle t', z_{(2)} \rangle ,\end{aligned}\tag{4.1}$$

---

<sup>6</sup>In the classical case  $\widehat{\beta}^{(+)}(I_A^{(0)}) = 1 \otimes I_A^{(0)}$ .

<sup>7</sup>Further we shall use the bicrossproduct from Sect. 3 with upper index (+), and subsequently drop this index.

where  $t, t' \in T_{4;2}^\kappa$  (generators  $P_\mu, \bar{Q}_\alpha$ ) and  $z, z' \in \bar{T}_{4;2}^\kappa$  (generators  $x^\mu, \bar{\theta}_\alpha$ ) and the orthonormal basis

$$\begin{aligned} \langle P_\mu, x^\nu \rangle &= \delta_\mu^\nu, & \langle \bar{Q}_{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}} \rangle &= i\delta_{\dot{\alpha}}^{\dot{\beta}}, \\ \langle \bar{Q}_{\dot{\alpha}}, x^\nu \rangle &= \langle P_\mu, \bar{\theta}^{\dot{\beta}} \rangle = 0, \end{aligned} \quad (4.2)$$

one can derive that

$$\begin{aligned} [x^0, x^k] &= -\frac{1}{\kappa}x^k, & [x^k, x^l] &= 0, \\ [x^0, \bar{\theta}^{\dot{\alpha}}] &= -\frac{1}{2\kappa}\bar{\theta}^{\dot{\alpha}}, & [x^\mu, \bar{\theta}^{\dot{\alpha}}] &= \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0, \end{aligned} \quad (4.3)$$

$$\begin{aligned} \Delta x_\mu &= x_\mu \otimes \mathbf{1} + \mathbf{1} \otimes x_\mu, \\ \Delta \bar{\theta}^{\dot{\alpha}} &= \bar{\theta}^{\dot{\alpha}} \otimes \mathbf{1} + \mathbf{1} \otimes \bar{\theta}^{\dot{\alpha}}. \end{aligned} \quad (4.4)$$

The relations (4.3 – 4.4) describe the chiral extension of  $\kappa$ -Minkowski space, introduced firstly by Zakrzewski [12]. The general formula describing the duality pairing can be written as follows (compare with [6]).

$$\begin{aligned} \langle f(P_i, P_0, \bar{\theta}_{\dot{\beta}}), : \psi(x^\mu, x^0, \bar{\theta}^{\dot{\alpha}}) : \rangle &= \\ = f\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^0}, i\frac{\partial}{\partial \bar{\theta}^{\dot{\beta}}}\right) \psi(x^\mu, x^0, \bar{\theta}^{\dot{\alpha}}) \Big|_{\substack{x^\mu=0 \\ \bar{\theta}^{\dot{\alpha}}=0}} \end{aligned} \quad (4.5)$$

where the normal ordering describes the functions of the  $\kappa$ -superspace coordinates with the generators  $x_0$  staying to the right from the generators  $x_i$ . The canonical action of  $\bar{T}_{4;2}^\kappa$  on the  $\kappa$ -deformed chiral superspace is given by the well-known formula

$$t \triangleright z = \langle t, z_{(1)} \rangle z_{(2)} \quad (4.6)$$

which gives  $P_\mu \triangleright x^\nu = \delta_\mu^\nu$  and

$$\begin{aligned} P_i \triangleright : \psi(x^i, x^0, \bar{\theta}^{\dot{\alpha}}) : &= : \frac{\partial}{\partial x_i} \psi(x^i, x^0, \bar{\theta}^{\dot{\alpha}}) :, \\ P_0 \triangleright : \psi(x^i, x^0, \bar{\theta}^{\dot{\alpha}}) : &= : \frac{\partial}{\partial x_0} \psi(x^i, x^0, \bar{\theta}^{\dot{\alpha}}) :, \\ \bar{Q}_{\dot{\alpha}} \triangleright : \psi(x^i, x^0, \bar{\theta}^{\dot{\alpha}}) : &= i : \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \psi(x^i, x^0, \bar{\theta}^{\dot{\alpha}}) :. \end{aligned} \quad (4.7)$$

In order to describe the action of  $\mathcal{U}(O(1, 3; 2))$  on  $\kappa$ -deformed superspace we dualize the action  $\alpha$  of  $\mathcal{U}(O(1, 3; 2))$  on Hopf superalgebra  $\bar{T}_{4;2}^\kappa$  (see (3.7)) by the well known formula:

$$\langle t, h \triangleright z \rangle = \langle \hat{\alpha}(h \otimes t), z \rangle. \quad (4.8)$$

We obtain the following covariant action of the generators of  $\mathcal{U}(O(1, 3; 2))$  on the  $\kappa$ -deformed chiral superspace

$$\begin{aligned}
M_i \triangleright x^0 &= 0, & M_i \triangleright x^j &= i\epsilon_{ijk}x^k, & M_i \triangleright \bar{\theta}^{\dot{\beta}} &= -\frac{1}{2}(\sigma_i)_{\gamma}^{\dot{\beta}}\bar{\theta}^{\gamma}, \\
N_i \triangleright x^0 &= -ix^i, & N_i \triangleright x^j &= -i\delta_i^j x^0, & N_i \triangleright \bar{\theta}^{\dot{\beta}} &= \frac{1}{2}(\sigma_i)^{\dot{\beta}}_{\gamma}\bar{\theta}^{\gamma}, \\
Q_{\alpha} \triangleright x^0 &= -2i\bar{\theta}^{\dot{\alpha}}, & Q_{\alpha} \triangleright x^j &= 2i(\sigma_j)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}, & Q_{\alpha} \triangleright \bar{\theta}^{\dot{\beta}} &= 0,
\end{aligned} \tag{4.9}$$

identical with the classical  $\kappa$ -independent action on covariant chiral superspace [13]. Further using the relation

$$x \triangleright (zz') = (-1)^{\eta(z)\eta(x_{(2)})}(x_{(1)} \triangleright z)(x_{(2)} \triangleright z'), \tag{4.10}$$

where  $x \in \mathcal{U}_{\kappa}(\mathcal{P}_{4;1})$ , one obtains the action of the  $\kappa$ -deformed  $N = 1$  Poincaré superalgebra generators on the functions of the  $\kappa$ -deformed chiral superspace coordinates. It can be checked that e.g. the action on the quadratic polynomials of  $(x^k, \bar{\theta}^{\alpha})$  contains anomalous  $\kappa$ -dependent terms.

The differential calculus on the  $\kappa$ -deformed chiral superspace and the theory of  $\kappa$ -deformed superfields will be described in our further publications. Such a calculus is described by the supersymmetric extension of the differential calculus on  $\kappa$ -Minkowski space (see [14, 15]) and it is different from the one described in [16] for quantum superspace with the generators satisfying quadratic algebra.

## Acknowledgements

One of the authors (J.L.) would like to thank Prof. L. Dabrowski and Prof. C. Reina for their hospitality at SISSA (Trieste), where the paper has been completed.

## References

- [1] J. Lukierski, A. Nowicki, H. Ruegg and V. Tolstoy, *Phys. Lett.* **B264**, 331 (1991)
- [2] S. Giller, J. Kunz, P. Kosiński, M. Majewski and P. Maślanka, *Phys. Lett.* **B286**, 57 (1992)
- [3] J. Lukierski, A. Nowicki and H. Ruegg, *Phys. Lett.* **B293**, 344 (1992)
- [4] J. Lukierski, A. Nowicki and J. Sobczyk, *J. Phys.* **A26**, L1109 (1993)
- [5] J. Lukierski, H. Ruegg and W. Rühl, *Phys. Lett.* **B313**, 357 (1993)
- [6] S. Majid and H. Ruegg, *Phys. Lett.* **B334**, 348 (1994)

- [7] P. Kosiński, J. Lukierski, P. Maślanka and J. Sobczyk, Wrocław University preprint IFTUWr 868/94, March 1994; *J. Phys. A*, in press
- [8] S. Majid, *Journ. Alg.* **130**, 17 (1990)
- [9] J. Lukierski, A. Nowicki and H. Ruegg, *Phys. Lett.* **B271**, 321 (1991)
- [10] S. Majid, *Foundations of Quantum Groups*, Chapt. 6, Cambridge University Press, in press
- [11] S. Majid, *Journ. Alg.* **163**, 165 (1994)
- [12] S. Zakrzewski, *Journ. of Phys*, **A27**, 2075 (1994)
- [13] S. Mandelstam, *Phys. Lett.* **121B**, 30 (1983)
- [14] A. Sitarz, *Noncommutative differential calculus on  $\kappa$ -Minkowski space*, DAMTP Cambridge University preprint, September 1994
- [15] S. Majid and H. Ruegg, in preparation
- [16] T. Kobayashi and T. Uematsu, Kyoto University preprint KUCP-47, May 1992.